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Quantum theory of rotation angles: The problem of angle sum and angle difference

A. Luis and L.L. Sánchez-Soto[a]

Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

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Abstract. We reconsider the problem of the sum and difference of two angle variables in quantum mechanics. The spectra of the sum and difference operators have widths of 4π, but angles differing by 2π are indistinguishable. This means that the angle sum and difference probability distributions must be cast into a 2π range. We obtain probability distributions for the angle sum and difference and relate this problem to the representation of nonbijective canonical transformations.

PACS. 03.65.Bz Foundations, theory of measurement, miscellaneous theories (including Aharonov Bohm effect, Bell inequalities, Berry's phase) – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

The proper definition of phase variables in quantum mechanics is beset by well-known difficulties (for reviews see Refs. [1–9]). For the simple case of a harmonic oscillator the problems essentially arise from two basic sources [10]: the periodicity and the semiboundedness of the energy spectrum. The first prevents the existence of a phase operator, but not of its exponential. The second entails that this exponential is not unitary [11–13].

In this paper, we shall rather focus on the similar problem of the description of the angular position of a plane rotator (henceforth referred to as an angle operator); *i.e.* a body in circular motion. Although we have the same kind of problems linked with the periodicity, for the plane rotator the angular momentum has a spectrum that includes both positive and negative integers. This allows to introduce a well behaved exponential of the angle operator [1].

We are primarily interested in the question of angle sum and difference, which is very similar to the equivalent problem for the phase [14–19]. It seems natural to define angle-sum and difference operators to be the sum and difference of the respective angle operators. However, due to its periodic character, adding and subtracting angles must be done carefully. Since each individual angle is expressed in a 2π range, the eigenvalue spectra of the sum and difference operators have widths of 4π, and this is not compatible with the idea that an angle variable must be 2π periodic. Thus, there should be a way to cast the angle sum and difference into the 2π range [20]. It is worth emphasizing that, although the probability distributions obtained using the ranges 4π and 2π are both valid, they give different values for the variances.

Our aim in this paper is to deduce a simple and adequate casting procedure for the problem at hand. We shall see that the transformation to the angle sum and difference is in fact nonbijective. After working out the consequences of this nonbijectivity, we reexamine the problem from the point of view of the canonical transformations, showing how the concept of ambiguity spin, introduced by Moshinsky and coworkers [21], fits in this context.

2 Classical rotation angles and simple quantization

We begin our discussion by reconsidering the problem of angular momentum in three dimensions. For simplicity, we restrict ourselves to a bead constrained to move on a circular wire whose axis is aligned in the Z direction. The classical azimuthal rotation angle of the bead can be defined in the window $[-\pi, \pi)$, for instance, as [22]

$$\phi(x, y) = 2 \arctan \left(\frac{\sqrt{x^2 + y^2}}{y} - \frac{x}{y} \right) \qquad y \neq 0, \qquad (2.1)$$

and for $y = 0$, ϕ is 0 or π according to $x > 0$ or $x < 0$, respectively. This exact definition, yet elementary, avoids the ambiguity associated with the π periodicity of the tangent function in the more *standard* definition $\phi(x, y) = \arctan(y/x)$.

The angle is defined as the inverse of a trigonometric function and may be defined to lie within a chosen 2π range or to be assigned an initial value and then evolve as

[a] e-mail: `lsanchez@eucmax.sim.ucm.es`

a continuous and unbounded variable [23]. If we treat ϕ as a continuous variable, then the Poisson bracket for the angle and the angular momentum has the form

$$\{\phi, L_z\} = 1. \tag{2.2}$$

Direct application of the correspondence between Poisson brackets and commutators, suggests the commutation relation (in units $\hbar = 1$)

$$[\phi, L_z] = i. \tag{2.3}$$

In the ϕ-representation, L_z can be represented by the differential operator

$$L_z = -i\frac{\partial}{\partial\phi}, \tag{2.4}$$

that verifies the fundamental relation (2.3). However, the use of this operator may entail many pitfalls for the unwary.

First, L_z given by (2.4) is selfadjoint only in the space of 2π-periodic functions. But ϕ itself is not periodic, and therefore if $\Psi(\phi)$ is a periodic wave function, then $\phi\Psi(\phi)$ is not periodic and is outside the angular momentum state space [23].

There is a further difficulty associated with a naive trust in the hermiticity of L_z. This problem was originally discovered in connection with the Dirac proposal of a phase operator [24]. Taking the matrix elements of (2.3) in the angular momentum basis we have

$$\langle m|[\phi, L_z]|m'\rangle = i\delta_{mm'}, \tag{2.5}$$

or, supposing that L_z can operate to the left as it were selfadjoint

$$(m' - m)\langle m|\phi|m'\rangle = i\delta_{mm'}. \tag{2.6}$$

The diagonal elements in this equation clearly demonstrates the problem.

A possible solution, proposed by Judge and Lewis [25], is to modify the angle operator so that it corresponds to multiplication by ϕ plus a series of step functions that sharply change the angle by 2π at appropriate points. The result is that the commutation relation between ϕ and L_z has a δ-function term in addition to the i term in (2.3). This corresponds to the classical Poisson bracket of L_z and a single-valued angle variable [23].

It is possible to follow a different method which seems simpler and gives the same results. The idea is to use a continuous periodic complex variable to locate the azimuthal position [26]. This was pointed out by Louisell [27] in the context of the phase problem. Thus we use the complex exponential of the angle, we shall denote by E, and impose the commutation relation

$$[E, L_z] = E. \tag{2.7}$$

The action of the unitary operator E on the angular-momentum basis is

$$E|m\rangle = |m-1\rangle, \tag{2.8}$$

where the integer m runs from $-\infty$ to $+\infty$. The eigenvectors of E are

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{+\infty} e^{im\phi}|m\rangle, \tag{2.9}$$

with $E|\phi\rangle = e^{i\phi}|\phi\rangle$, and allow for a resolution of the identity of the form

$$I = \int_{\phi_0}^{\phi_0+2\pi} d\phi\; |\phi\rangle\langle\phi| \tag{2.10}$$

where ϕ_0 is a fiducial or reference angle [23]. This family of projectors, and the associated probability distribution $P(\phi) = \langle\phi|\rho|\phi\rangle$, where ρ is the density operator for the system, represents an ideal, sharp or noiseless description of the angle.

The resolution of the identity (2.10) allows us to introduce an angle operator

$$\Phi_{\phi_0} = \int_{\phi_0}^{\phi_0+2\pi} d\phi\; \phi|\phi\rangle\langle\phi|. \tag{2.11}$$

The properties and proper use of this operator have been studied extensively by Barnett and Pegg. The interesting approach developed in reference [23] involves the use of a state space of finite but arbitrarily large dimension where the angle eigenstates can be properly normalized. Physical results are obtained in the limit as the dimension tends to infinity after expectation values are calculated.

Although the vectors $|\phi\rangle$ provide an adequate description of the quantum angle, it should be taken into account that realistic measurements are always imprecise. In particular, the measurement of $P(\phi)$ would require infinite energy. In other words, the mathematical continuum of angles will be observed always with finite resolution [28].

Therefore, it could be interesting to extend the quantum angle formalism by including fuzzy, unsharp or noisy generalizations of the ideal description provided by E or $|\phi\rangle\langle\phi|$. To this end we shall use positive-operator measures (POMs) [29,30], that are a set of linear operators $\Delta(\phi)$ furnishing the correct probabilities in any measurement process through the fundamental postulate that

$$P(\phi) = \mathrm{Tr}\left[\rho\Delta(\phi)\right]. \tag{2.12}$$

Compatibility with the properties of ordinary probability imposes the requirements

$$\Delta^\dagger(\phi) = \Delta(\phi), \qquad \Delta(\phi) \geq 0, \qquad \int_{\phi_0}^{\phi_0+2\pi} d\phi\Delta(\phi) = I. \tag{2.13}$$

One important point is, however, that the operators $\Delta(\phi)$ might be nonorthogonal projections and mixed states.

In addition to these basic statistical conditions, some other requirements must be imposed to ensure that $\Delta(\phi)$ provides a meaningful description of the angle as a canonically conjugate variable with respect L_z (even in the sense of a weak Weyl relation). To this end, we adopt the same axiomatic approach developed previously by Leonhard, Vaccaro, Böhmer and Paul [31] for the optical phase. First, we require the shifting property [32]

$$e^{i\phi' L_z} \ \Delta(\phi) \ e^{-i\phi' L_z} = \Delta(\phi + \phi'), \tag{2.14}$$

which leads to

$$\Delta(\phi) = \frac{1}{2\pi} \sum_{m,m'=-\infty}^{\infty} g_{m,m'} \ e^{i(m-m')\phi} |m\rangle\langle m'|. \tag{2.15}$$

We must take also into account that a shift in L_z should not change the angle distribution. A shift in L_z is expressed by the operator E since, according to (2.8), it shifts the angular momentum distribution by one step. Therefore, we require as well

$$E \ \Delta(\phi) \ E^\dagger = \Delta(\phi), \tag{2.16}$$

which imposes the additional constraint $g_{m+1,m'+1} = g_{m,m'}$. This means that

$$g_{m,m'} = g_{m-m'}. \tag{2.17}$$

In consequence, (2.15) can be recast as

$$\Delta(\phi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} g_{-k} \ e^{-ik\phi} E^k, \tag{2.18}$$

and the conditions (2.13) are now

$$|g_k| \le 1, \qquad g_k^* = g_{-k}. \tag{2.19}$$

Expressing the operator E in terms of its eigenvectors (2.9), we finally arrive at the more general form of the POM describing the angle variable and fulfilling the natural requirements (2.14) and (2.16):

$$\Delta(\phi) = \int_{\phi_0}^{\phi_0+2\pi} d\phi' \ G(\phi') \ |\phi+\phi'\rangle\langle\phi+\phi'|, \tag{2.20}$$

where

$$G(\phi) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} g_k \ e^{-ik\phi}. \tag{2.21}$$

The convolution in (2.20) shows that this POM effectively represents a noisy version of the usual projection measure $|\phi\rangle\langle\phi|$. The function $G(\phi)$ gives the resolution provided by this POM.

3 Probability distributions for the angle sum and difference

When we have two of such plane rotators, labeled 1 and 2, the exponentials of the angle sum E_+ and angle difference E_- are the unitary operators

$$E_+ = E_1 E_2 \ , \qquad E_- = E_1 E_2^\dagger. \tag{3.1}$$

We introduce as well the angular momentum sum and difference by

$$L_+ = \frac{L_{1z}+L_{2z}}{2} \ , \qquad L_- = \frac{L_{1z}-L_{2z}}{2}. \tag{3.2}$$

These operators satisfy the commutation relations

$$\begin{aligned} &[E_-, L_+] = 0, \qquad [E_+, L_+] = E_+, \\ &[E_-, L_-] = E_-, \qquad [E_+, L_-] = 0; \end{aligned} \tag{3.3}$$

so they are in fact canonically conjugate variables. The eigenvectors of E_+ and E_- are of the form $|\phi_1,\phi_2\rangle$ with eigenvalues $e^{i\phi_+} = e^{i(\phi_1+\phi_2)}$ and $e^{i\phi_-} = e^{i(\phi_1-\phi_2)}$, respectively.

Note that while (E_1,E_2), (L_{1z},L_{2z}) or (L_+,L_-) are complete sets of commuting operators, this is not true for (E_+,E_-), since the vectors $|\phi_1,\phi_2\rangle$ and $|\phi_1+\pi,\phi_2+\pi\rangle$ have the same angle sum and difference. Therefore another commuting operator must be considered to describe the system. We can use the operator

$$V = e^{i\pi(L_{1z}+L_{2z})}, \tag{3.4}$$

which commutes with E_+ and E_-

$$[E_+, V] = [E_-, V] = 0. \tag{3.5}$$

In consequence, (E_+,E_-,V) is a complete set of commuting operators, whose associated basis is

$$|\phi_+,\phi_-,p\rangle = \frac{e^{-ip\phi_1}}{2} \left[|\phi_1,\phi_2\rangle + (-1)^p |\phi_1+\pi,\phi_2+\pi\rangle \right], \tag{3.6}$$

with $p = 0, 1$, and

$$\phi_1 = \frac{\phi_+ + \phi_-}{2} \ , \qquad \phi_2 = \frac{\phi_+ - \phi_-}{2}. \tag{3.7}$$

The complex exponential in the definition (3.6) is introduced for convenience in order to get the same expression $|\phi_+,\phi_-,p\rangle$ when ϕ_1 and ϕ_2 are replaced by $\phi_1+\pi$ and $\phi_2+\pi$. Then, the action of V on this basis is

$$V|\phi_+,\phi_-,p\rangle = (-1)^p |\phi_+,\phi_-,p\rangle, \tag{3.8}$$

and we have the resolution of the identity

$$I = \sum_p \int_{\phi_{0+}}^{\phi_{0+}+2\pi} \int_{\phi_{0-}}^{\phi_{0-}+2\pi} d\phi_+ d\phi_- \ |\phi_+,\phi_-,p\rangle\langle\phi_+,\phi_-,p| \ , \tag{3.9}$$

where ϕ_{0+} and ϕ_{0-} are the corresponding fiducial angles for these variables.

The joint-probability distribution function $\mathcal{P}$ cast into a 2π range for the angle sum and difference associated with a system state ρ is

$$\mathcal{P}(\phi_+,\phi_-) = \sum_{p=0,1} \langle\phi_+,\phi_-,p|\rho|\phi_+,\phi_-,p\rangle, \tag{3.10}$$

which is the sum of the contributions from each value of p.

Taking into account (3.6) and (3.7), we can express $\mathcal{P}(\phi_+, \phi_-)$ in terms of the probability distribution for the individual angles $P(\phi_1, \phi_2) = \langle \phi_1, \phi_2 | \rho | \phi_1, \phi_2 \rangle$ in the form

$$\mathcal{P}(\phi_+, \phi_-) = \frac{1}{2}\left[P\left(\frac{\phi_+ + \phi_-}{2}, \frac{\phi_+ - \phi_-}{2}\right) + P\left(\frac{\phi_+ + \phi_-}{2} + \pi, \frac{\phi_+ - \phi_-}{2} + \pi\right)\right]. \tag{3.11}$$

An equivalent way to obtain this law is to compute the characteristics [33] $e^{ik\phi_+}\, e^{il\phi_-}$, with k and l integers:

$$\left\langle e^{ik\phi_+}\, e^{il\phi_-}\right\rangle = \int_{\phi_{01}}^{\phi_{01}+2\pi}\int_{\phi_{02}}^{\phi_{02}+2\pi} d\phi_1 d\phi_2 \times e^{ik(\phi_1+\phi_2)} e^{il(\phi_1-\phi_2)}\, P(\phi_1, \phi_2). \tag{3.12}$$

We must get the same mean values for any periodic function of the angle sum and difference whether we use the variables (ϕ_+, ϕ_-) or (ϕ_1, ϕ_2), and then

$$\int_{\phi_{0+}}^{\phi_{0+}+2\pi}\int_{\phi_{0-}}^{\phi_{0-}+2\pi} d\phi_+ d\phi_-\; e^{ik\phi_+} e^{il\phi_-} \mathcal{P}(\phi_+, \phi_-)$$

$$= \int_{\phi_{01}}^{\phi_{01}+2\pi}\int_{\phi_{02}}^{\phi_{02}+2\pi} d\phi_1 d\phi_2\; e^{ik(\phi_1+\phi_2)} e^{il(\phi_1-\phi_2)} P(\phi_1, \phi_2). \tag{3.13}$$

Since $\mathcal{P}(\phi_+, \phi_-)$ and $P(\phi_1, \phi_2)$ are 2π-periodic functions, these equalities determine $\mathcal{P}(\phi_+, \phi_-)$ completely, as can be shown using Fourier analysis.

We see that the probability distribution for the angle sum and difference cannot be obtained from the one associated with the individual angles simply by the corresponding transformation of the variables (3.7). This is because the same sum and difference can be obtained from two different values for the angles of each system and then the transformation becomes nonbijective. The true transformation is obtained only after adding these two contributions.

We can generalize now the transformation law (3.11) to any POM. The joint-probability distribution function $P(\phi_1, \phi_2)$ will arise from $\Delta(\phi_1, \phi_2)$ defined by

$$\Delta(\phi_1, \phi_2) = \Delta_1(\phi_1) \otimes \Delta_2(\phi_2). \tag{3.14}$$

The use of (3.12) leads to the following POM for the phase sum and difference cast into a 2π range

$$\Lambda(\phi_+, \phi_-) = \frac{1}{2}\left[\Delta\left(\frac{\phi_+ + \phi_-}{2}, \frac{\phi_+ - \phi_-}{2}\right) + \Delta\left(\frac{\phi_+ + \phi_-}{2} + \pi, \frac{\phi_+ - \phi_-}{2} + \pi\right)\right]. \tag{3.15}$$

When Δ_1 and Δ_2 are of the form (2.18) we have

$$\begin{aligned}\Lambda(\phi_+, \phi_-) &= \frac{1}{(2\pi)^2}\sum_{k,l=-\infty}^{\infty} g^{(1)}_{-k-l}\; g^{(2)}_{-k+l} \times e^{-ik\phi_+} e^{-il\phi_-}\; E_+^k E_-^l \\ &= \sum_p \int_{\phi_{0+}}^{\phi_{0+}+2\pi}\int_{\phi_{0-}}^{\phi_{0-}+2\pi} d\phi'_+ d\phi'_-\; G(\phi'_+, \phi'_-) \\ &\quad\times |\phi_+ + \phi'_+, \phi_- + \phi'_-, p\rangle\langle \phi_+ + \phi'_+, \phi_- + \phi'_-, p|,\end{aligned} \tag{3.16}$$

with

$$G(\phi_+, \phi_-) = \frac{1}{(2\pi)^2}\sum_{k,l=-\infty}^{\infty} g^{(1)}_{-k-l}\; g^{(2)}_{-k+l}\; e^{ik\phi_+} e^{il\phi_-}, \tag{3.17}$$

and the relation between $G(\phi_+, \phi_-)$ and $G_1(\phi_1)G_2(\phi_2)$ is given by (3.11).

As it could be expected, $\Lambda(\phi_+, \phi_-)$ is a fuzzy generalization in the form of a convolution of the ideal description of angle sum and difference in (3.10). This POM satisfies the natural requirements (2.14) and (2.16) with respect to L_+ and L_-. Moreover, it does not contain any information about the variable p.

Finally, we focus on the angle difference. The associated POM $\Lambda(\phi_-)$ is defined by

$$\Lambda(\phi_-) = \int_{\phi_{0+}}^{\phi_{0+}+2\pi} d\phi_+\; \Lambda(\phi_+, \phi_-). \tag{3.18}$$

By using (3.16), we get

$$\Lambda(\phi_-) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} g^{(1)}_{-k}\; g^{(2)}_{k}\; e^{-ik\phi_-}\; E_-^k. \tag{3.19}$$

This is equivalent to

$$\Lambda(\phi_-) = \int_{\phi_{02}}^{\phi_{02}+2\pi} d\phi'\, \Delta_1(\phi_- + \phi')\Delta_2(\phi'). \tag{3.20}$$

This last equation allows us to provide an alternative approach to the fuzzy descriptions of angle discussed in Section 2. If we consider that the density operator factorizes, $\rho = \rho_1 \otimes \rho_2$, the angle difference can be regarded as a measure of the angle ϕ_1 relative to a given reference state described by ρ_2.

If $\Delta_1(\phi_1) = |\phi_1\rangle\langle\phi_1|$ and $\Delta_2(\phi_2) = |\phi_2\rangle\langle\phi_2|$, the angle description of the system ρ_1 that equation (3.20) provides in this way is

$$\begin{aligned}\Delta(\phi) &= \mathrm{Tr}_2\left[\rho_2\; \Lambda(\phi = \phi_-)\right] \\ &= \int_{\phi_{02}}^{\phi_{02}+2\pi} d\phi' \langle\phi'|\rho_2|\phi'\rangle |\phi + \phi'\rangle\langle\phi + \phi'|,\end{aligned} \tag{3.21}$$

and now the function $G(\phi)$ is $\langle\phi|\rho_2|\phi\rangle$. Thus, any POM with the properties (2.14) and (2.16) can be viewed as an ideal measure of the angle relative to an imprecise origin of angles described by the reference state ρ_2.

4 Canonical transformation to angle sum and difference

In this section we wish to reconsider the transformation relating the set of coordinates $(\phi_1, \phi_2, L_{1z}, L_{2z})$ of the phase space of the system to the set $(\phi_+, \phi_-, L_+, L_-)$. This is a canonical transformation; *i.e.* it preserves the Poisson brackets and thereby (ϕ_+, L_+) and (ϕ_-, L_-) are conjugate variables.

It is clear that a similar transformation in position and momentum for instance will not need any special caution. Even if the range of variation of the position would be a finite interval, we could always accommodate properly the range of variation of the sum and difference variables. However, in the angle case we are forced to think on ϕ_+ and ϕ_- as 2π-periodic variables. This necessary restriction makes the transformation nonbijective since, as previously discussed, the points (ϕ_1, ϕ_2) and $(\phi_1 + \pi, \phi_2 + \pi)$ map on the same point (ϕ_+, ϕ_-).

The equivalence between phase-space coordinates related by a canonical transformation is expressed in quantum mechanics by a unitary transformation [34]. This means that now we have two Hilbert spaces $\overline{\mathcal{H}}_+$ and $\overline{\mathcal{H}}_-$ associated with angle and angular momentum operators $(\bar{E}_+, \bar{L}_+)$ and $(\bar{E}_-, \bar{L}_-)$ related to the original ones $\mathcal{H}_1$ and $\mathcal{H}_2$, associated with $(E_1,\ L_{1z})$ and $(E_2,\ L_{2z})$, *via* a unitary operator $U : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \overline{\mathcal{H}}_+ \otimes \overline{\mathcal{H}}_-$, such that

$$\begin{aligned} \bar{E}_+ &= U E_+ U^\dagger \ , \ \bar{E}_- = U E_- U^\dagger \ , \\ \bar{L}_+ &= U L_+ U^\dagger \ , \ \bar{L}_- = U L_- U^\dagger \ . \end{aligned} \tag{4.1}$$

The knowledge of U provides complete information about the transformation we are studying.

Intimately linked with the nonbijectivity, we find that the transformation must relate operators with different spectra [21]. Relations (4.1) seem to impose half-integer values to $\bar{L}_+$ and $\bar{L}_-$ (that is, 4π-periodicity for the corresponding angles) contrary to what we have supposed. Therefore, the transformation (4.1) cannot be unitary.

Nevertheless we can find isometric mappings [35] if we restrict the definition to certain subspaces of $\mathcal{H}_1 \otimes \mathcal{H}_2$. This can be accomplished by using the concept of ambiguity group; *i.e.* the group connecting the set of points in the original space mapped on the same one in the new space [21]. Here this group has only two elements: the identity and a joint π rotation on both angles, that is represented by the operator V in (3.4). Note that this group is equivalent to the cyclic group of order 2, and leaves invariant all the operators in the definition (4.1) of the transformation.

To find subspaces that could be isometrically mapped in $\overline{\mathcal{H}}_+ \otimes \overline{\mathcal{H}}_-$, verifying (4.1) up to constants, we must restrict ourselves to subspaces where the action of the ambiguity group becomes a constant phase factor; *i.e.* the subspaces carrying the unitary representations of the group. According to (3.8), we have two of these subspaces, we shall call $\mathcal{E}_0$ and $\mathcal{E}_1$, spanned by $|\phi_+, \phi_-, p\rangle$ $(p = 0, 1)$. Equivalently, the subspace $\mathcal{E}_0$ is spanned by the simultaneous eigenvectors of L_{1z} and L_{2z} $\{|2n, 2m\rangle, |2n{+}1, 2m{+}1\rangle\}$, while $\mathcal{E}_1$ is spanned by $\{|2n+1, 2m\rangle, |2n, 2m+1\rangle\}$, with n and m integers running from $-\infty$ to $+\infty$. Note that in avoiding the nonbijectivity with these restrictions, we also remove the problem caused by the difference of the spectra. The subspace $\mathcal{E}_0$ has only eigenvalues of L_{1z} and L_{2z} whose sum or difference is even, and then the spectra of the operators involved in (4.1) are equal. On the other hand, $\mathcal{E}_1$ contains only eigenvalues whose sum or difference is odd, and the spectra can be made equal simply adding to $\bar{L}_+$ and $\bar{L}_-$ in (4.1) a half-integer constant. With this in mind, it is possible to find two isometric mappings U_p from $\mathcal{E}_p$ $(p = 0, 1)$ to $\overline{\mathcal{H}}_+ \otimes \overline{\mathcal{H}}_-$. They are given by

$$U_p = \int_{\phi_{0+}}^{\phi_{0+}+2\pi} \int_{\phi_{0-}}^{\phi_{0-}+2\pi} d\phi_+ d\phi_- \ |\bar{\phi}_+, \bar{\phi}_-\rangle\langle \phi_+, \phi_-, p| \ , \tag{4.2}$$

where

$$|\bar{\phi}_+, \bar{\phi}_-\rangle = \frac{1}{2\pi} \sum_{m,n=-\infty}^{\infty} e^{im\bar{\phi}_+} e^{in\bar{\phi}_-} |\bar{m}, \bar{n}\rangle \tag{4.3}$$

are the common eigenstates of $\bar{E}_+$ and $\bar{E}_-$ and $|\bar{m}, \bar{n}\rangle$ are the eigenstates of $\bar{L}_+$ and $\bar{L}_-$. We can observe that $U_p U_p^\dagger = I$ while $U_p^\dagger U_p$ is the projector on the subspace $\mathcal{E}_p$. Moreover, $\sum_p U_p^\dagger U_p = I$.

With these partial isometries we can construct a unit-preserving completely positive map [36,37] $\varUpsilon : \overline{\mathcal{H}}_+ \otimes \overline{\mathcal{H}}_- \to \mathcal{H}_1 \otimes \mathcal{H}_2$, such that for any operator A

$$\varUpsilon(A) = \sum_p U_p^\dagger \ A \ U_p. \tag{4.4}$$

In particular, we have

$$\varUpsilon(\bar{E}_-) = E_1 E_2^\dagger, \qquad \varUpsilon(\bar{E}_+) = E_1 E_2. \tag{4.5}$$

Despite this, we could be interested in a truly unitary transformation defined over the whole space, in order to have a complete description of the system in terms of the angle sum and difference. To do this we need to enlarge the final space adding a new variable, usually called the ambiguity spin, whose role is to provide a different image for each subspace $\mathcal{E}_p$ and simultaneously equalize the spectra. The final space will be of the form $\overline{\mathcal{H}}_+ \otimes \overline{\mathcal{H}}_- \otimes \mathcal{V}$, where $\mathcal{V}$ is a two-dimensional Hilbert space spanned by the orthonormal basis $|0\rangle$ and $|1\rangle$.

Considering $U : \mathcal{H}_1 \otimes \mathcal{H}_2 \longrightarrow \overline{\mathcal{H}}_+ \otimes \overline{\mathcal{H}}_- \otimes \mathcal{V}$, we have that the operator

$$U = |0\rangle U_0 + |1\rangle U_1, \tag{4.6}$$

is unitary and performs the transformation

$$\begin{aligned} \bar{E}_+ &= U E_+ U^\dagger \ , \quad \bar{E}_- = U E_- U^\dagger \ , \\ \bar{L}_+ + \frac{\varPi}{2} &= U L_+ U^\dagger \ , \ \bar{L}_- + \frac{\varPi}{2} = U L_- U^\dagger \ , \end{aligned} \tag{4.7}$$

where $\bar{\Pi}|p\rangle = p|p\rangle$.

The unitary operator U contains the transformation (4.4) as a particular case, since

$$U^\dagger \ (A \otimes I_{\mathcal{V}})\, U = \sum_p U_p^\dagger \ A \ U_p, \tag{4.8}$$

$I_{\mathcal{V}}$ being the identity in $\mathcal{V}$.

With this unitary transformation we immediately obtain the probability distribution function associated with the angle sum and difference. One easily checks that

$$\begin{aligned}\mathcal{P}(\phi_+,\phi_-) &= \mathrm{Tr}\left[\rho U^\dagger \ \left(|\bar{\phi}_+,\bar{\phi}_-\rangle\langle\bar{\phi}_+,\bar{\phi}_-| \otimes I_{\mathcal{V}}\right) U\right] \\ &= \sum_{p=0,1} \langle\phi_+,\phi_-,p|\rho|\phi_+,\phi_-,p\rangle. \end{aligned} \tag{4.9}$$

Thus, once again, we have arrived at (3.10) and to the same transformation law (3.11) by a different way. The nonbijectivity, ambiguity group and ambiguity spin translate the fact that the angle sum and difference are not by themselves a complete set of commuting operators. Then, the ambiguity spin is the other operator needed to complete this set. Note that $e^{i\pi\bar{\Pi}} = U \ V \ U^\dagger$.

5 Conclusions

What we expect to have accomplished in this paper is a complete description of the probability distribution for the angle sum and difference. We have shown that this probability distribution, when cast to a 2π interval, has a nontrivial form if expressed in terms of the probabilities of individual angles. This is due to the fact that the same sum and difference can be obtained from two different values of the angles of each system.

This fact makes the transformation from individual angles to angle sum and difference nonbijective. We have worked the problem from the point of view of the representations of nonbijective canonical transformations, adapting the concept of the ambiguity spin to this context.

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